Renormalization group and operator product expansion in turbulence: Shell models

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(Received 24 March 1993)

A general role of the renormalization group (RG) in the theory of fully developed turbulence is proposed, with the simple case of the shell models as an illustrative example. A Wilson-type RG is defined, i.e., a transformation in a space of shell-dynamics "subgrid models" with fixed uv cutoff, for a class of theories with fixed mean dissipation and strength of quadratic nonlinearity. It is explained that, if a zero-viscosity limit exists, then its "subgrid" dynamics below the cutoff is necessarily (near) a fixed point of the RG transformation. Conversely, any RG fixed-point subgrid model is associated to a zero-viscosity limit. By means of an "asymptotic completeness" assumption for the fixed point, a high shell-number expansion is established, analogous to the operator product expansion (OPE) of field theory. This expansion predicts characteristic "multifractal scaling" for shell variable moments and also relations between inertial and dissipation range scaling exponents. Furthermore, under the plausible assumption of an "additive OPE," a predicted scaling form for two-point moment correlations is established. The results of this paper are nonperturbative but only of a qualitative character, based upon precise assumptions about the fixed-point theory. However, we also discuss the possibility of an implementation of RG by numerical methods (Monte Carlo, decimation, etc.) or perturbation expansion to test the assumptions and to make a quantitative evaluation of the scaling exponents. The relation of RG to naive "cascade ansatz" is also discussed.

PACS number(s): 47.27.Ak, 64.60.Ak

I. INTRODUCTION

In many papers analogies between scaling in high-Reynolds-number turbulence and equilibrium critical phenomenon have been proposed [1], and attempts have been made to apply the powerful *renormalization-group* (RG) methods, which have proved so effective and illuminating in critical phenomenon, to turbulence scaling [1–3]. However, to our knowledge, none of these works has really provided a logic for such an application of RG to turbulence, nor emphasized the proper goals of such an RG study. We shall attempt here to supply the missing foundations of the RG method in turbulence. A brief account of our approach has appeared in [4], but we would like in this work to give a more detailed discussion with an emphasis on basic principles. Particularly, we would like to emphasize the relation of RG to simple dimensional analysis (DA) and to explain in what sense it allows one to extend and improve the results of DA. In general, going beyond DA requires certain properties of the RG fixed point which, however, are often mild, plausible, or expected to be generic. As illustrations of this potential of RG, we derive some simple qualitative predictions of RG for model problems which cannot be obtained from DA alone.

To prevent any confusion, let us state at the outset that our approach is entirely distinct from the ϵ-expansion "renormalization-group" (RNG) theory of Yakhot and Orszag [3]. Their approach is based on a set of approximations and implicit assumptions independent of standard renormalization-group methodology, and there is no basis in a systematic RG theory for the claimed asymptotic validity of their results. We discuss the ϵ expansion and some of its problems briefly in the last section of this paper. Our analysis here is entirely nonperturbative, based on a set of explicit assumptions which are stated as clearly and precisely as possible. Subject to validity of the assumptions, our results are exact.

For the most part in this work, we shall confine our attention to so-called *shell models* of turbulence. These were originally introduced as radically simplified models of the Navier-Stokes equation by Gledzer [5] and Desnyanski and Novikov [6]. Later, Siggia [7] (see also Zimin [8]) gave a more careful qualitative "derivation" of the shell-type models from the Navier-Stokes equation, in the spirit of Wilson's phase-space cell analysis leading to the approximate recursion formula [9]. In agreement with Siggia's original arguments and despite the simplicity of the dynamics, the shell models have been found by direct numerical simulation to exhibit strong deviations from the classical Kolmogorov scaling behavior predicted by a dimensional analysis [10]. We discuss here primarily the shell models because, as we emphasize, the obvious application of RG is to a high-Reynolds-number limiting behavior. Although high-Reynolds-number flows abound in nature, there are great difficulties in practice in making precision measurements of short-distance statistics in the laboratory or in the field. With the present experimental techniques, the use of incompletely founded assumptions like the "Taylor frozen-turbulence hypothesis," with an unknown range of validity, are necessary to relate experimental measurements to theoretical quantities. It is not clear that the experiments are wholly reliable even for qualitative checks on theory. On the other hand, direct numerical simulation of the Navier-Stokes equation is without such ambiguities, but present computational resources limit us to the study of rather low-Reynolds-number flows, say, $\mathrm{Re}\sim 100$. Such a Reynolds number is too low to exhibit a meaningful scaling range. The shell models, in contrast, because of the radically reduced number of degrees of freedom, can be simulated at extremely high effective Reynolds number (e.g., $\mathrm{Re}\sim 2^{16}$), exhibit wide scaling ranges, and are nevertheless rich enough to exhibit deviations from Kolmogorov's mean-field predictions. The models are, in fact, despite the simplifications, still formidably difficult at the theoretical level and retain some of the chief difficulties of the original problem: essential strong nonlinear coupling of modes and a high degree of statistical disequilibrium. Although our discussion is mostly made in the context of the shell models, most of it carries over more or less directly to the realistic fluid equations. We shall point out the general features in the detailed discussion below.

The contents of this paper are as follows. In Sec. I we define the shell models in a precise way and also describe the steady-state statistical problem with external driving forces. We consider the simplest model problem of a Gaussian random force, white noise in time, which provides a fixed average rate of energy injection in low shells. Afterwards we make a complete dimensional analysis for this problem. In Sec. II we define the natural RG transformation for this problem, which is completely determined by the requirements to keep fixed (1) the strength of quadratic nonlinearity $\lambda\equiv 1$ and (2) the mean value of energy dissipation (more properly, energy injection rate) $\bar{\epsilon}$. As we explain, the fixed-point condition under this RG is more or less equivalent to the existence of a well-defined zero-viscosity limit. A qualitative picture of the RG flow is sketched, based on physically reasonable ideas, and it is emphasized that turbulence scaling arises in an important way from its *near-critical* character. Then some technical methodology of RG theory is introduced: the notion of scaling variables and the relation of linearized RG transformations to correlation functions at the fixed point. In Sec. III we derive the qualitative predictions of RG theory for shell-model turbulence scaling: in particular, "multifractal"-type scaling for shell variable moments and asymptotic scaling laws for joint correlations of shell moments. The basic assumptions made are a "completeness" assumption for eigenvariables of the RG map and an "additive-coupling" assumption which is strongly motivated from the physics. The fundamental tool is a high-shell-number expansion, which is derived from RG theory on the basis of the assumptions. We also discuss relationships of RG with simpler "cascade ansatz," which are intuitive but *ad hoc,* that have played a leading role previously in determining the form of corrections to Kolmogorov scaling. Finally, in Sec. IV, we briefly discuss the possibility of quantitative calculation of scaling exponents with RG methods, implemented by numerical means, e.g., Kraichnan's decimation ideas [11], or perturbation expansion.

The background on RG theory required for this work can be adequately provided by Wilson's article [12] and by Wegner's very strong technical discussion in [13]. For turbulence theorists, Wilson's article is especially recommended as a good conceptual introduction to renormalization-group ideas and methods. (See particularly his intuitive discussion in the Introduction of the relation of RG to the "cascade picture" in critical phenomena.) Another illuminating presentation of RG, emphasizing its generality and expressing a philosophy very close to ours, is contained in the book of Goldenfeld [14]. One of the important general points for statistical mechanics we would like to make in this work is the broad applicability in principle of RG methods to *nonequilibrium problems* with a wide range of excited scales. Important features of RG theory of second-order phase transitions are connected to the fact that the RG map there acts in a space of local effective Hamiltonians. However, RG theory is too often presented in a form which is essentially tied to that framework, whereas the basic concepts and methods actually apply in a much more general context.

II. SHELL MODELS

A. Definition of the models

As we indicated in the Introduction, the shell models were originally introduced as model problems with some essential similarities to more realistic fluid equations [5,6]. Later Siggia, in particular, made a more careful attempt to justify these models [7]. In other words, without any pretense of developing a quantitative approximation to the Navier-Stokes equation, Siggia argued that the shell dynamics are nevertheless a good qualitative model of high-Reynolds-number fluid behavior. In particular, he argued that the local interactions (in wave number, or scale)—which are the only interactions retained in the shell dynamics—are the most essential set of interactions both in transport of energy in scale and in producing deviations from classical Kolmogorov scaling for the Navier-Stokes equation [7]. (See also [15] on the issue of local energy transfer.) The shell models have no spatial degrees of freedom and correspond roughly to a hierarchically arranged sequence of eddies in the original problem, each eddy containing one or a fixed small number of eddies of size half as large. The important feature is the *nonproliferation* of modes, each dyadic wavenumber shell $2^n k_0 \le |\mathbf{k}| < 2^{n+1}k_0$, $k_0=1/L$ containing the same fixed number of modes as $n\to\infty$. In contrast, in a true d-dimensional fluid problem the number of eddies of size $2^{-n}L$ increases as $\sim 2^{dn}$. Hence the shell models are properly to be thought of as zero dimensional. The essential mechanism of deviations from Kolmogorov scaling in the shell models was suggested by Siggia to be a buildup of (temporal) intermittency in the local cascade, resulting in a "bursting" behavior of energy flux. Siggia indeed observed this behavior in his earliest simulations [16]. This is in line with the original Landau argument against Kolmogorov's 1941 theory (K41) [17], that it neglected the fluctuation in the dissipation rate (in that sense, K41 is "mean field"). It is likely that the intermittency at equal Reynolds numbers is more severe for the shell variables than for corresponding quantities in the Navier-Stokes case (like wavelet amplitudes), because there is no diffusive transport of energy in the shell model "sideways in space," i.e., to other eddies of the same scale, which can act to smooth out the energy transfer in the real fluids [18].

We do not here make any use of explicit forms of the shell dynamics, but we do indicate the general form and certain required properties for a qualitative approximation to the fluid equations. The basic structure of the shell models is that they are a set of ordinary differential equations (possibly infinite dimensional), quadratically nonlinear and first order in time, for a set of "shell-variables" u_n, where n is a discrete index running over some interval of integers $[M,N]$. The variable u_n may be real or complex, with a possible finite degeneracy $u_{n\alpha}$, indexed by α: for simplicity, we discuss mostly the real, nondegenerate case as an example. The variable u_n is to be thought of as a mode at "wave number" $k_n \equiv 2^n k_0$. The general form of the dynamics is therefore

$$(\partial_t + \nu k_n^2) u_n(t) = \sum_{m,l} A_{nml} u_m(t) u_l(t) + f_n(t) , \tag{1}$$

with ν a "viscosity." In principle, other dissipative mechanisms might be employed, e.g., a "hyperviscosity" term ηk_n^4. The term f_n is a driving force to produce a dissipative steady state, which we discuss more below. The nonlinear coupling A_{nml} should satisfy a number of requirements. First, it should be "local," i.e., be only finite range or even nearest neighbor. Second, $A_{nlm} = k_n R_{l-n,m-n}$, to represent that the nonlinearity is proportional to wave number. Also, it should exhibit "energy conservation," i.e., the quantity

$$E(t) = \tfrac{1}{2} \sum_n u_n^2(t) , \tag{2}$$

should be a formal constant of the motion with $\nu = 0$. A sufficient condition for this is the "detailed conservation condition"

$$A_{nml} + A_{mln} + A_{lnm} = 0 , \tag{3}$$

with the symmetry $A_{nml} = A_{nlm}$. Furthermore, the dynamics should satisfy a "Liouville theorem" when $\nu = 0$,

$$\sum_n \frac{\partial \dot{u}_n}{\partial u_n} = 0 . \tag{4}$$

The latter two conditions guarantee that the Gaussian "Gibbs distributions"

$$d\mu[u] = \frac{1}{Z} \prod_n du_n \exp\left[-\beta \sum_n u_n^2 / 2 \right] \tag{5}$$

will be stationary measures for the inviscid dynamics. The above properties are easy to engineer.

Some other requirements are also important, although it is not necessarily clear how to design models with the requisite behavior *a priori.* For example, one wants the model to have a good ergodic behavior, both for the inviscid case and also for the driven, damped case. The latter is clearly important from the point of view of the Ruelle-Takens conception of turbulence, as being associated to a strange attractor for the dissipative dynamics [19]. Furthermore, good ergodic properties for the free, inviscid dynamics implies a tendency toward the energy equipartition exhibited by the equilibrium measure in Eq. (5). This has implications even for the driven, damped case, since it indicates energy injected in the low shells, for example, will have a tendency to flow out to higher shells under the nonlinear dynamics. In conjunction with this, it is reasonable to require that "energy" $E(t)$ as defined in Eq. (2) should be the only extensive invariant of the inviscid dynamics: otherwise, there shall be invariant subsets of the energy shell, persisting in the large-N limit, and additional "equilibrium measures" besides the equipartition distribution Eq. (5) (however, this would be a desirable property for shell models of two-dimensional turbulence.) Although such properties are hard to build in, there are some known cases which are numerically observed to have such features. A currently popular complex-variable version, the "Okhitani-Yamada model," is an example [20–22].

B. Dissipative cascade states

To produce a steady-state problem, we use the common device of an external driving field f_n which acts only in some low shells, in a finite range around $n=0$, as a source of energy. These forces might be deterministic or random. Especially convenient is the choice of a Gaussian force with zero mean and covariance

$$\langle f_n(t) f_m(t') \rangle = F(k_n) \delta_{n,m} \delta(t-t') . \tag{6}$$

If there is a steady state for this forcing, then it is a simple consequence of the Gaussian integration-by-parts identity that the mean dissipation in that steady state, i.e.,

$$\bar{\epsilon} \equiv \sum_n \nu k_n^2 \langle u_n^2 \rangle , \tag{7}$$

must be related to the noise covariance as

$$\bar{\epsilon} = \tfrac{1}{2} \sum_n F(k_n) \tag{8}$$

(e.g., see Novikov [23]). This is an exact "energy-balance" relation for the steady-state, expressing that the mean rate of energy injection must be equal to the mean rate of dissipation. The only way it might fail is if the system is underdamped (too small viscosity for a given N) and no steady state exists at all. However, one might reasonably expect that the system is adequately damped for *any* $\nu > 0$ when $N = +\infty$. In that case, there are always sufficiently high shell numbers n such that νk_n^2 is as large as desired, for any given $\nu > 0$. This produces a curious circumstance that the result Eq. (8) should be valid for arbitrary small ν, and, since the right-hand side is *independent* of ν, the limit as $\nu \to 0$ of the left-hand side is strongly suggested to be the right-hand side, a finite, positive value. This might appear paradoxical, since the dynamics formally conserves energy for $\nu = 0$. However, a remark made originally by Onsager [24] (see also [15]) for the true Navier-Stokes dynamics applies also here: the inviscid dynamics, when $N = +\infty$, does not necessarily conserve energy, even when "detailed conservation" Eq. (3) holds. Mathematically, the problem is that detailed conservation only implies overall conservation if the infinite-series expression for dE/dt is unconditionally convergent and may be arbitrarily reordered. That requires that the series obey an absolute summability requirement. It is easy to check, using $A_{nml}=O(2^n)$, that the series is indeed absolutely convergent when the shell variables obey a growth condition

$$|u_n|=O(2^{-nh}) \tag{9}$$

for $h>1/3$. (This is the exact analog for the shell variables of Onsager's requirement of Hölder index $h>\frac{1}{3}$ for the fluid velocities.) If $h\le\frac{1}{3}$, then energy need not be conserved. The physics of Onsager's observation is that energy may be lost in the infinite cascade to $N=+\infty$. Indeed, most of the familiar shell models have an *exact* stationary solution for the bi-infinite shell array, $M=-\infty, N=+\infty$, of the form

$$u_n^{(0)}\equiv C\bar{\epsilon}^{1/3}k_n^{-1/3}\ , \tag{10}$$

which has a constant flux of energy to (or from) $N=+\infty$. It is therefore not dynamically incomprehensible that a *dissipative* zero-viscosity limit may exist yet be governed by the inviscid shell-model equations.

C. Dimensional analysis

Our model is now completely defined. At this stage, it is useful to consider a dimensional analysis to see what are the permitted forms of various statistical averages purely on those simple grounds. The dynamical equations contain three dimensional constants: k_0, the wavenumber scale, which particularly for forcing just near $n=0$ represents an inverse stirring length L^{-1}; the (kinematic) viscosity ν; and an overall normalization factor of the noise strength F_0, which has, by Eq. (8), the same dimensions as mean rate of dissipation of energy (per mass) $\bar{\epsilon}$. The latter quantities are essentially identified. In units of length $[L]$ and time $[T]$ the above quantities have dimensions as $[k_0]=[L^{-1}]$, $[\nu]=[L^2T^{-1}]$, and $[\bar{\epsilon}]=[L^2T^{-3}]$. The shell variables themselves are readily determined to have the dimension of a velocity $[u_n]=[LT^{-1}]$. Let us consider the stationary statistical state of the shell dynamics with $M=0$ and finite N. Then for a typical average, like a shell variable moment $\langle u_n^q\rangle$, dimensional analysis requires a form

$$\langle u_n^q\rangle_{\bar{\epsilon},\nu,N,L}=\bar{\epsilon}^{q/3}k_n^{-q/3}A^{(q)}\left[k_nL,k_NL,\frac{\bar{\epsilon}}{\nu^3k_N^4}\right]\ , \tag{11}$$

with $A^{(q)}$ an undetermined function. The dimensionless combination $g^2\equiv\bar{\epsilon}/\nu^3k_N^4$ has a natural significance. If one defines rescaled quantities with dimensions powers of length *only* in such a way as to set $\nu=1$, $\bar{\epsilon}=1$, i.e., $t'=\nu t$, $u_n'=(\nu/\bar{\epsilon})^{1/2}u_n$, $f_n'=(\bar{\epsilon}\nu)^{-1/2}f_n$, then the nonlinear term in the shell dynamics appears with a proportionality $\bar{\epsilon}/\nu^3$. Hence g is a dimensionless measure of the effective nonlinearity. (For this reason, it is sometimes defined as $g^2\equiv\bar{\epsilon}\lambda^2/\nu^3k_N^4$, where λ is a *formal* nonlinearity strength actually set $\lambda\equiv1$.)

The first Kolmogorov similarity hypothesis, in our context, is essentially that the limit of both $L\to+\infty$ and $N\to+\infty$ should exist for correlation functions, like simple moments. Then, DA yields

$$\langle u_n^q\rangle_{\bar{\epsilon},\nu}=\bar{\epsilon}^{q/3}k_n^{-q/3}B^{(q)}(\bar{\epsilon}/\nu^3k_n^4)\ . \tag{12}$$

This is usually written with the dimensionless combination inside the scaling function taken to be $k_n\eta$, where $\eta=\nu^{3/4}/\bar{\epsilon}^{1/4}$ is the "inner" or "dissipation" length scale. The second similarity hypothesis was that the further limit $\nu\to0$ should exist. In that case, the expectations $\langle u_n^q\rangle$ will be functions in the inertial range of $\bar{\epsilon}$ *only* out of all the dimensional constants as well as of k_n, so that DA gives then

$$\langle u_n^q\rangle_{\bar{\epsilon}}=C_q\bar{\epsilon}^{q/3}k_n^{-q/3}\ , \tag{13}$$

with C_q dimensionless constants. We therefore sometimes refer to $d_1=-\frac{1}{3}$ as the *canonical dimension* of u_n. It is indeed the engineering dimension when u_n is converted to length units by $\bar{\epsilon}$ only: $\bar{u}_n=u_n/\bar{\epsilon}^{1/3}$. Because of Landau's argument, it is dubious that Kolmogorov's assumption of a finite $L\to+\infty$ limit for correlations is valid, but, as discussed above, the zero-viscosity limit is on much better footing. A generalized form of the Kolmogorov similarity hypotheses, taking into account Landau's criticism, is still to assume existence of the limits $N\to+\infty$ and subsequently $\nu\to0$ but to allow for a dependence also on stirring-length L, for which DA then yields the result

$$\langle u_n^q\rangle_{\bar{\epsilon},L,\nu}=\bar{\epsilon}^{q/3}k_n^{-q/3}E^{(q)}\left[k_nL,\frac{\bar{\epsilon}}{\nu^3k_n^4}\right]\ , \tag{14}$$

and in the inertial-range limit $\nu\to0$,

$$\langle u_n^q\rangle_{\bar{\epsilon},L}=\bar{\epsilon}^{q/3}k_n^{-q/3}F^{(q)}(k_nL)\ , \tag{15}$$

with $E^{(q)},F^{(q)}$ undetermined scaling forms.

III. THE RENORMALIZATION GROUP

A. The space of theories and definition of the renormalization-group transformation

The Wilson-type RG transformation for our class of problems can be defined as a map in a space of *large-scale effective dynamics* or *subgrid-scale eddy models.* The idea is that the ensemble of histories of the low-shell-number modes below a uv cutoff N, generated by the *full* dynamical problem with $N=+\infty$ (or very large), may be taken as the sample paths of an abstractly defined random process of the low-shell variables. Even if the original shell dynamics, as defined above, was of Langevin-type, the low-shell effective dynamics need not be. In principle the effective dynamics may be defined as follows: construct a Feynman path-integral representation for the generating functionals of the full process, corresponding to a Martin-Siggia-Rose (MSR) field theory as in [25,26]. For our shell-model case, this representation has the form

$$Z[h,\hat{h}]=\int \prod_{n,t} du_n(t)d\hat{u}_n(t)\exp\left\{\sum_n \int_{-\infty}^{+\infty}dt\left[i\hat{u}_n(t)\left[\partial_t u_n+\nu k_n^2 u_n-\sum_{m,l}A_{nml}u_m(t)u_l(t)\right]\right.\right.$$
$$\left.\left.-\tfrac{1}{2}F(k_n)\hat{u}_n^2(t)+u_n(t)h_n(t)+\hat{u}_n(t)\hat{h}_n(t)\right]\right\}, \qquad (16)$$

where $Z[h,\hat{h}]$ is the generating functional whose functional derivatives with respect to the sources $h_n(t)$ generate correlation functions of the $u_n(t)$. [The derivatives with respect to the sources $\hat{h}_n(t)$ generate response functions, but we do not need this for the present discussion. Also, we may note that the Liouville theorem was used to get rid of a Jacobian factor.] Then, simply integrating out of the path integral the high-shell variables $(n>N)$ gives a path-integral representation (of highly complicated form) for the generating functional of the low-shell process. If the original dynamics had a high-k_n cutoff as well (e.g., the shell dynamics is truncated at a k_Λ, the molecular scale), then there is no difficulty in principle in defining the path integrals and performing (abstractly) the elimination procedure. The path integral of the truncated model is defined in terms of ordinary integrals if the dynamics is discretized in time. To help regularize the integrations one may add to every shell a weak Gaussian white noise—which may be thought to represent molecular noise—and then, after final computation of averages, set it equal to zero. [This is also the proper procedure to select the physical Sinai-Bowen-Ruelle (SBR) measure on the Ruelle-Takens attractor.] The practical computational problem of performing the elimination averages will be severe in the absence of a small expansion parameter.

Nevertheless, for the construction of the RG map the above definition suffices. As always, the Wilson RG map acts in a space of theories *with fixed uv cutoff N*. It is convenient to take the lowest shell number to $M=-\infty$: the model may be then ir regularized in various ways, e.g., by forcing only shell variables near $n=0$ and strongly damping the modes for $n<0$. Also, it is often clarifying (although a little pedantic) to use below the probabilists' notation of capitals $U_1,U_2,\ldots$ to denote abstract random variables and lower case $u_1,u_2,\ldots$ to denote values assumed, with some probability, in particular realizations of the ensemble. After eliminating the highest variable U_N from the collection $\{\ldots,U_0,\ldots,U_N\}$ and performing a set of suitable rescalings which, in particular, map $U_{n-1}\to U'_n$, one ends up with an effective dynamics of the variables $\{\ldots,U'_1,\ldots,U'_N\}$, again a theory with uv cutoff N. The specific rescalings which are performed depend upon the physics of the problem under consideration. In our case, we want to understand the high-k_n scaling which is observed in the limit $\nu\to 0$, with fixed $\bar{\epsilon}$ and L (and, in particular, to establish the existence of such a state). Therefore, we naturally require the RG to act in a space of theories with fixed $\bar{\epsilon}$. Note that we *cannot* require that L be fixed since the rescaling required to bring the uv cutoff back to N necessarily transforms $L\to L/2$. Another natural requirement is that λ stay fixed (at 1): this corresponds to the imposition of Galilei covariance for the real fluid case. There is a simplifying feature of the RG for the shell dynamics which is worth noting in this context. Because of the *strict locality* of the shell dynamics, the elimination procedure only modifies the MSR Lagrangian in the finite interaction range below N. Therefore, all of the complicated features of the subgrid model (non-Markovian behavior, higher-order nonlinearity, etc.) pile up just below the cutoff and the dynamics beyond a finite range below the cutoff remains the original shell-model dynamics. This allows us to prescribe in a completely unambiguous way the rescalings necessary to keep $\bar{\epsilon}$ and λ fixed. Indeed, consider the set of possible rescalings:

$$k'_{n+1}=k_n,\quad t'=2^{-z}t,\quad u'_{n+1}(t')=2^x u_n(t)\ . \qquad (17)$$

Making the substitutions in the original (low-shell) dynamics, one finds easily that the rescaled equation is

$$(\partial_{t'}+\nu' k'^2_n)u'_n(t')=2^{z-1-x}\sum_{m,l}A_{nml}u'_m(t')u'_l(t')+f'_n(t') \qquad (18)$$

with $\nu'=2^{z-2}\nu$ and $f'_n(t')=2^{z+x}f_n(t)$. Hence $\lambda'=\lambda=1$ requires $z=1+x$, while $\bar{\epsilon}'=\bar{\epsilon}$, or fixed noise strength, requires $z+2x=0$. [Observe that it is really the energy injection rate, given by Eq. (8), which is being held fixed.] Therefore, we see that we are forced by our prescriptions to take $x=-\frac{1}{3}$ and $z=\frac{2}{3}$. It is worth noting that x appears as a scaling dimension of the shell variables and that $-\frac{1}{3}$ is the *canonical dimension* as discussed above.

Although the above RG procedure is essentially dynamical, it immediately yields a corresponding "static" RG by specializing to only single-time distributions of the shell variables. The RG map of low-shell probability distribution functions (PDF's) is defined by $P_{\{U\}}\to P'_{\{U\}}\equiv P_{\{U'\}}$. We shall actually discuss mostly throughout this paper just this resulting static RG. The dynamic RG contains further interesting physical information, but the essential qualitative features of the RG flow behavior should be identical—in the shell-model case—for dynamics as for statics. Therefore, we restrict ourselves to statics for simplicity.

B. Eddy viscosity and the meaning of renormalization-group invariance

The resulting rescaling of the viscosity is as $\nu'=2^{-4/3}\nu$. Observe that under successive rescalings the viscosity in the shells far below N goes rapidly to zero: after k rescalings $\nu\to 2^{-4k/3}\nu$. However, this is not the case for the shells in the interaction range just below the cutoff N. The elimination step will produce new contributions to the effective viscosity of those cutoff scale modes resulting from the average energy drain of the eliminated shell degrees of freedom just above it. This is the so-called "eddy viscosity" (which, in the shell model, has the peculiarity to be a strictly cutoff scale effect, through lack of nonlocal shell interactions.) Let the total effective viscosity at the cutoff scale after k elimination steps (before any rescalings) be $\nu^{(N-k)}$. (It corresponds to a cutoff shell number $N-k$.) Then the *RG-transformed viscosity* (rescaled effective viscosity after k steps) is $\nu_k^{(N)}=2^{-4k/3}\nu^{(N-k)}$. Let us note the following obvious but very important conclusion: a fixed point dynamics can exist under the RG with fixed $\bar{\epsilon}$ and λ, and in particular, a fixed point effective viscosity $\nu_*^{(N)}$, if and only if $\nu_*^{(N)}\sim\bar{\epsilon}^{1/3}k_N^{-4/3}$. The importance of this conclusion is seen particularly by considering its equivalent for the realistic fluid model: a fixed-point state invariant under Euler dynamics with a finite mean dissipation $\bar{\epsilon}$ must have an effective viscosity at a momentum scale k of the form $\nu(k)\sim\bar{\epsilon}^{1/3}k^{-4/3}$. The same conclusion can be reached in another way, by considering the requirement that the dimensionless coupling $g=(\bar{\epsilon}/\nu^3k_N^4)^{1/2}$ should approach a fixed point under successive RG transformations, $g_k\to g_*$. Clearly, $\nu_*^{(N)}=(\bar{\epsilon}/g_*^2)^{1/3}k_N^{-4/3}$.

The criterion for a fixed point (FP) requires some care in its definition, since, as long as $L\neq 0,\infty$, a strict FP cannot exist, for L is halved under iteration. Here we take as the "approximation fixed-point condition" the following: let $P_{U_1,\ldots,U_N}(u_1,\ldots,u_N;\bar{\epsilon},L/2)$ be the joint PDF of the shell variables $U_1,\ldots,U_N$ in the state with stirring length $L/2$ (with uv cutoff N, as always) and $P_{U_1',\ldots,U_N'}(u_1,\ldots,u_N;\bar{\epsilon},L)$ be the joint PDF of the *RG-transformed variables* $U_1',\ldots,U_N'$ in the state with stirring length L. (These PDF's are the stationary distributions of the corresponding effective dynamics.) Then we require as the FP condition that

$$P_{U_1',\ldots,U_N'}(u_1,\ldots,u_N;\bar{\epsilon},L)=P_{U_1,\ldots,U_N}(u_1,\ldots,u_N;\bar{\epsilon},L/2)\ . \tag{19}$$

This exactly corresponds to the "approximate FP condition" proposed by Wilson in numerical study of finite-size critical systems [27]. We could just as well put here PDF's for the full set of shell variables ranging to $M=-\infty$. However, it is actually the $L\to\infty$ limit of the PDF (assumed to exist) which is the fixed point of the RG, so that the variables for $k_n<L^{-1}$ are not really of concern. This has an exact analog in the theory of critical phenomena, where the FP theory is necessarily a *critical theory*, i.e., the correlation length $\xi^*=+\infty$. The reasoning in that context is exactly the same, since ξ is halved in each RG step and the only fixed points of $\xi'=\xi/2$ are $\xi^*=+\infty$ (or $\xi^*=0$, which is generally not realized.) We would like to emphasize that the existence of the $L\to+\infty$ limit, or "Kolmogorov fixed point," for effective PDF's or subgrid dynamics is one of the fundamental assumptions of our approach. As we shall discuss below, the $L\to+\infty$ limit in turbulence is likely to be rather singular, and while it may reasonably be expected to exist at the level of PDF's, the resulting distributions are probably *momentless,* i.e., with extremely long tails that lead to infinite moments of all orders. Furthermore, real turbulent systems are never precisely at the fixed point, and this leads to some characteristic features of turbulence scaling.

A point which does not seem to be appreciated in the literature is that the RG fixed-point condition is nothing but a formal restatement of the fact that the theory is independent of the molecular viscosity (however, see [1], Sec. II). It follows indeed directly by dimensional analysis from that assumption. In fact, under the condition that the PDF is a function only of $\bar{\epsilon}$ and L, it has the following form by DA:

$$P_{U_1,\ldots,U_N}(u_1,\ldots,u_N;\bar{\epsilon},L)=(\bar{\epsilon}L)^{-N/3}f(u_1/(\bar{\epsilon}L)^{1/3},\ldots,u_N/(\bar{\epsilon}L)^{1/3};k_1L,\ldots,k_NL)\ . \tag{20}$$

Now, it can be verified at once from this form that

$$\begin{aligned}P_{U_1',\ldots,U_N'}(u_1,\ldots,u_N;\bar{\epsilon},L)&=2^{N/3}P_{U_0,\ldots,U_{N-1}}(2^{1/3}u_1,\ldots,2^{1/3}u_N;\bar{\epsilon},L)\\&=2^{N/3}(\bar{\epsilon}L)^{-N/3}f(2^{1/3}u_1/(\bar{\epsilon}L)^{1/3},\ldots,2^{1/3}u_N/(\bar{\epsilon}L)^{1/3};k_0L,\ldots,k_{N-1}L)\\&=\left[\bar{\epsilon}\frac{L}{2}\right]^{-N/3}f\left[u_1\Big/\left[\bar{\epsilon}\frac{L}{2}\right]^{1/3},\ldots,u_N\Big/\left[\bar{\epsilon}\frac{L}{2}\right]^{1/3};k_1\frac{L}{2},\ldots,k_N\frac{L}{2}\right]\\&=P_{U_1,\ldots,U_N}(u_1,\ldots,u_N;\bar{\epsilon},L/2)\ .\end{aligned} \tag{21}$$

In particular, any zero-viscosity limit, if it exists, has automatically distributions of the subgridscale shell variables which are (near) an RG fixed point. This is the exact counterpart of the situation in field theory where the renormalization-group invariance is just a restatement of the fact that the theory is independent of the uv cutoff, e.g., a lattice size. [The terms "RG equation" or "RG invariance condition" are really more appropriate for our Eq. (19), and the term "fixed-point condition" should be reserved for the limiting case $L\to+\infty$.] The triviality of the deduction should be appreciated. It may not be obvious why this formal restatement of the independence should even be useful. One answer to this is that there is a converse statement to the above which is rather less trivial: namely if there is a fixed point in the space of theories with fixed cutoff N, then it follows that a continuum ($N=+\infty$) zero-viscosity limit may be constructed with high-k_n behavior determined by this fixed point. For the RG considered above this fixed point will be governed by the $\lambda=1$ shell dynamics and have the fixed mean dissipation $\bar{\varepsilon}>0$. The procedure for this construction is described by Wilson at length in Sec. 12.2 of [28], and we shall discuss it more briefly just below. Here we shall simply assume the existence of the zero-viscosity limit for PDF's, as a second fundamental assumption of our approach. Aside from the more foundational issue of existence of the limit, there are also practical advantages to the RG statement of viscosity independence. As we shall discuss in detail in the following, some modest assumptions on the fixed point allow one to *supercede dimensional analysis* and derive qualitative results which cannot be derived by DA alone. Furthermore, the RG technology can also be applied to yield calculational strategies for deriving quantitative results on the predicted scaling behavior, e.g., anomalous exponents.

C. Renormalization-group flow, physical scaling regimes, and renormalization

It is perhaps helpful at this point to give a more pictoral description of the RG flow and its relation to physical cascade states of the shell model. A hypothetical rendering of the RG flow is indicated in Fig. 1. The dotted line indicates a particular RG flow trajectory. Following the RG flow corresponds to progressing up in scale through a turbulent cascade state, passing through effective theories for k_n scales going from higher to lower values (although each is rescaled back to fixed cutoff k_N). The lines with given "integral" or "outer" scale L are indicated. Note that L decreases by factors of $\frac{1}{2}$ in each RG iteration. Furthermore, a fixed point, labeled F, is indicated on the "critical line" $L=+\infty$. Consider a cascade state at an extremely high Reynolds number. For k_n much larger than the Kolomogorov scale $k_\eta=\bar{\varepsilon}^{1/4}\nu^{-3/4}$, the effective viscosity $\nu^{(n)}$ is still essentially the same as the original (bare) viscosity. (If there is another dissipation mechanism, such as a hyperviscosity, then a different criterion than $k_n \gg k_\eta$ will apply, but there will still be a regime in which the dissipation is dominated by the "bare" mechanism rather than the eddy viscosity.) This is the so-called "dissipation range" and corresponds to effective theories with large L but still far from the fixed point F. Now, as one moves up in scale the eddy viscosity begins to increase, from the effect of all the eliminated degrees of freedom, and the fixed point is gradually approached. The range of scales for which the RG phase point is near the fixed point is characterized by the scaling behavior associated to that fixed point and represents the so-called "inertial (sub)range." If L was sufficiently large at the outset (the Reynolds number was sufficiently high), then the RG phase point will spend a very long "time" in the vicinity of the fixed point. However, eventually L will begin to become small under RG iteration and the phase point will leave the neighborhood of the fixed point. Then one has entered the "inertial superrange" or "energy range," in which, in fact, most of the energy of the state is contained. This part of the RG flow is, of course, highly nonuniversal if the usual ideas of a local cascade—with statistically independent successive steps, leading to loss of large-scale information—are correct. In that case there will be *very many* small-k_n states, corresponding to different stirring or generation mechanisms, which have the same high-k_n statistics, determined by the same RG fixed point. In this regard, Fig. 1 does not qualitatively well represent the RG flow: in fact, there should be an infinite-dimensional space of theories and an infinite number of RG trajectories running out of the fixed point to different small-k_n behavior. This is roughly represented in Fig. 2. In the language of RG theory, the different flowlines leaving the fixed point are referred to as *relevant directions,* and we see that the fixed point corresponding to the high-Reynolds-number scaling behavior should have infinitely many such ir unstable directions.

Let us note here briefly that each of the RG trajectories emanating from the fixed point (relevant directions) corresponds to a possible *zero-viscosity limiting theory.* The different theories corresponding to different trajec-

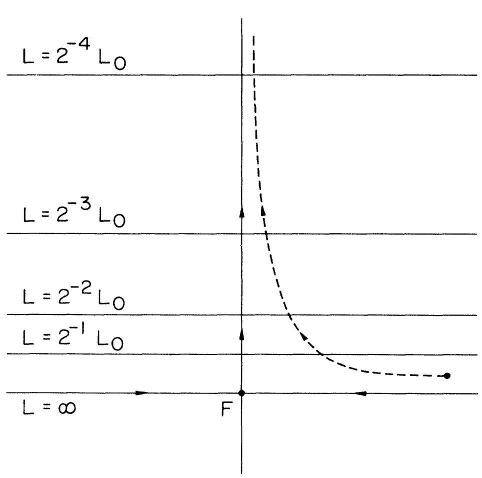

FIG. 1. Renormalization-group trajectory.

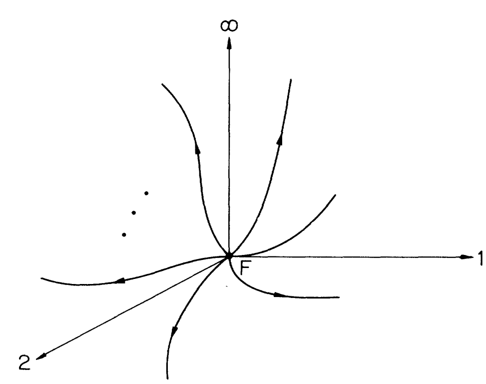

FIG. 2. Schematic representation of the RG flow.

tories have different large-scale behavior but the same uv scaling behavior determined by the fixed point. These are *continuum theories,* i.e., no uv cutoff, so it is meaningful in principle to talk about high-k_n scaling. The method of constructing the theories is discussed at length by Wilson and Kogut in Sec. 12.2 of [28]: the argument uses certain *scaling relations* near the fixed point, which we shall discuss further below. However, graphically it is easy to see how a sequence of cutoff theories, with cutoff k_N, may be selected so that as $N \to +\infty$, the RG trajectory of the cutoff theory converges to the ir unstable trajectory out of the fixed point. In Fig. 3 we indicate such a sequence, and the convergence to the trajectory of the relevant direction (only one is indicated) is obvious from the topology of the RG flow. In fact, with (infinitely) many relevant directions there will be a multitude of possible "renormalization procedures" producing different limiting theories, and the selection of any one such theory will require the careful fine tuning of a large (infinite) number of parameters in the "bare sequence." We will say no more here about these issues.

For now, we introduce a little more of the technical RG apparatus that we need to derive some of the promised qualitative predictions. The main concepts we want to introduce are those of *scaling variables* and associated scaling dimensions; also, the idea of the *linearized RG map* and, particularly, its relation with the correlation functions at the fixed point. The concept of scaling variables (and associated "scaling fields") was originally developed in order to discuss *near-critical* scaling in equilibrium phase transitions (see Wegner [29]). That is, it was introduced to allow description of effects due to a finite correlation length ξ, and it generalized naturally Wilson's idea of "eigenoperators" of the linearized map at the fixed point. As we have already emphasized, the near criticality of turbulence scaling is quite fundamental, so these concepts are naturally required in our context. The following material is at times rather technical, so those with previous unfamiliarity may wish to read the rest of this section rather quickly, for the basic idea, and return for a closer study after seeing the specific applications in the following section.

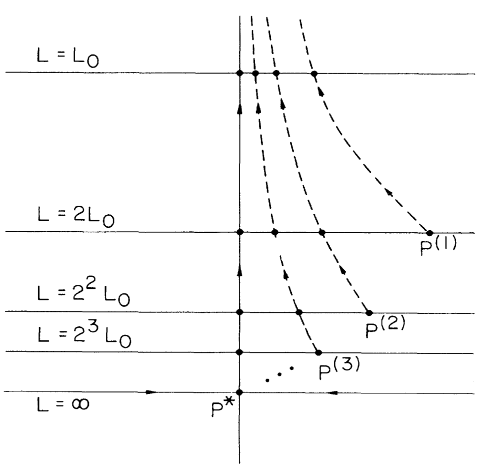

FIG. 3. A renormalization sequence.

D. Renormalization-group analysis of general perturbations

Let us first discuss the topic of renormalization group extended to general random variables (or "composite operators") of the system. To begin, it is useful to note a reformulation of the RG map $P_{\{U\}} \to P'_{\{U\}}$ of effective distributions in terms of the *generating functionals*

$$W[h] = \left\langle \exp\left[i \sum_n U_n h_n \right] \right\rangle , \tag{22}$$

where $\langle\ \rangle$ denotes expectation with respect to the distribution $P_{\{U\}}$ of the stochastic variables U for the effective theory. Define "transformed sources" as

$$h'_n = \begin{cases} 2^{-1/3} h_{n+1}, & n \le N-1 \\ 0, & n = N . \end{cases} \tag{23}$$

Then, if $W'[h]$ is the generating functional of the one-step RG-transformed distribution, it follows from the definitions that $W'[h] = W[h']$ since $\sum_n h'_n U_n = \sum_n h_n U'_n$. If one has a variable $S[U]$, which is some functional of the shell variables $\{U_n : n \in [-\infty, N]\}$, then one can write down a generating functional including the variable S as a supplemental (nonautonomous) variable, formally as

$$W[h,\kappa] = \exp\left[\kappa S \left[\frac{\delta}{i\delta h} \right] \right] W[h] , \tag{24}$$

where κ is the source variable corresponding to $S[u]$. Equivalently,

$$W[h,\kappa] = \left\langle \exp\left[i \sum_n U_n h_n + \kappa S[U] \right] \right\rangle . \tag{25}$$

Likewise, the joint generating functional of a collection of such variables $\{S_\alpha\}$ may be defined as

$$W[h,\kappa] = \left\langle \exp\left[i \sum_n U_n h_n + \sum_\alpha \kappa_\alpha S_\alpha[U] \right] \right\rangle . \tag{26}$$

Let us write for the single variable case $W[h,S] \equiv W[h,1] = \langle \exp(i \sum_n U_n h_n + S[U]) \rangle$. We are assuming in our discussion that the S_α's (and therefore the sources κ_α, too) are dimensionless.

The definition of the "renormalization-group map" $\mathcal{R}$: $S \to S'$ uses the intuitive idea that any real-valued $S[u]$ may be considered as a "perturbation" of the effective distribution $P_{\{U\}}$, by taking

$$W^\kappa[h] \equiv W[h,\kappa] \tag{27}$$

as the definition of a generating functional for a new distribution $P^\kappa_{\{U\}}$. (Write $P^S_{\{U\}}$ for $\kappa=1$.) This is well-defined subject to an integrability condition on $\exp[\kappa S[U]]$. Notice, however, that this perturbed distribution is in general *unnormalized,* i.e., it is not a probability measure. The normalization factor is just $1/W[0,\kappa]$. For this perturbation the RG-transformed distribution, i.e., the distribution of the RG-transformed variables U' under the perturbed distribution, $(P^S_{\{U\}})' \equiv P^S_{\{U'\}}$, defines an "RG-transformed variable" $S'[U]$, which is the logarithm of the density of that measures with respect to $P'_{\{U\}}$. In other words,

$$W[h',S]=W'[h,\mathcal{R}\{S\}] \tag{28}$$

or

$$\left\langle \exp\left[i\sum_n U_n h_n' + S[U]\right]\right\rangle = \left\langle \exp\left[i\sum_n U_n h_n + S'[U]\right]\right\rangle' . \tag{29}$$

Notice that the transformed perturbed distribution is compared to what would have been the transformed distribution in the absence of the perturbation. It is worth emphasizing here that the only requirement for defining the map $\mathcal{R}$ is the rather weak one of existence of densities. Even in the case where the measure on phase space has fractal support, as in the Ruelle-Takens scenario, the physical or SBR measure on the attractor is smooth along unstable manifolds [30]. This should be enough to guarantee smoothness of the reduced distributions, which is all that RG deals with. A direct consequence of the definition is the equation

$$W[0,S]=W'[0,\mathcal{R}\{S\}] , \tag{30}$$

which can be interpreted to say that S and $\mathcal{R}\{S\}$ have identical normalization factors considered as perturbations of $P_{\{U\}}$ and $P'_{\{U\}}$, respectively.

The whole subject is based on making a useful interplay between the different interpretations of the two sides of Eq. (27), the left as the generating functional of a "perturbed" distribution and the right as a joint generating function for the primitive variables U and the dependent variable $S[U]$. Because $\mathcal{R}$ is a nonlinear map, $\mathcal{R}\{\kappa S\}(U)=S'(U;\kappa)$ will ordinarily be a nonlinear function of the sources. It may be evaluated as a power series in κ by expanding both sides of Eq. (29) (with $S\to\kappa S$) in the source κ and equating terms of like order. However, because $S'(U;0)=0$ by its definition, the first nonvanishing term in this expansion must be at least linear in κ,

$$S'[U;\kappa]=\kappa\mathcal{T}\{S\}[U]+O(\kappa^2) , \tag{31}$$

where $\mathcal{T}$: $S[U]\to S'[U]$ is easily checked to be a linear map. This map depends, of course, upon the starting effective distribution $P_{\{U\}}$ in the above discussion (it is just the tangent or differential map to $\mathcal{R}$ at that point.) Because of the equation Eq. (31), $\langle \exp[i\sum_n U_n h_n + S'[U;\kappa]]\rangle'$ may still be usefully regarded as a joint generating functional for arbitrary correlations of the U's and at most one insertion of $S'[U]$ in the state $P'_{\{U\}}$. If there is a subspace of variables invariant under $\mathcal{T}$, which is spanned by an independent set $\{S_\alpha\colon \alpha\in\mathcal{J}\}$, then we may introduce a matrix of $\mathcal{T}$ in that subspace, as

$$\mathcal{T}\{S_\alpha\}=\sum_{\beta\in\mathcal{J}} T_{\alpha,\beta}S_\beta . \tag{32}$$

For the joint generating function of the variables $\{S_\alpha\colon \alpha\in\mathcal{J}\}$ it follows from Eqs. (28) and (31) that to linear order in each of the sources κ_α,

$$W[h',\kappa]=W'[h,\kappa'] , \tag{33}$$

where

$$\kappa'_\beta=\sum_{\alpha\in\mathcal{J}}\kappa_\alpha T_{\alpha,\beta} \tag{34}$$

is the linearized RG map of the sources. The functional $W'[h,\kappa']$ may be legitimately used to generate correlation functions for $P'_{\{U\}}$ with at most one insertion of each of the variables $S'_\alpha[U]$, by differentiating with respect to the original κ's.

E. Linearized transformations and scaling variables

In the case where the starting distribution was a fixed point $P^*_{\{U\}}$, the map $\mathcal{T}^*$ is called the *linearized renormalization-group map* at the fixed point. It is a fundamental assumption of most RG applications, which must be verified in each particular case, that near each fixed point there is a complete set of variables $\{O_\beta\colon \beta\in\mathcal{J}\}$ (within some class) which are *eigenvectors* of the linear map, i.e.,

$$\mathcal{T}_*\{O_\alpha\}=2^{-x_\alpha}O_\alpha . \tag{35}$$

This property is sometimes referred to as *asymptotic completeness,* since it is supposed to be true as the RG phase point approaches sufficiently close to the fixed point. The eigenvalue exponent x_α is the so-called *(Wilson) scaling dimension of the variable* O_α. The origin of the term "scaling dimension" will appear just below. When considered as perturbations of the fixed-point distribution, the variables with $x_\alpha>0$ are decaying under the RG iteration and are termed *irrelevant,* while the variables with $x_\alpha<0$ are growing perturbations and are termed *relevant.* The variables with $x_n=0$ are a special class, termed *marginal.* It should be mentioned that it is generally the codimension in space, $y_\alpha=d-x_\alpha$, which would appear in Eq. (35), but in the shell-model case $y_\alpha=-x_\alpha$, since $d=0$. Whereas it is usually the inequality $x_\alpha<d$, which is the condition for relevancy, for the shell-model existence of classes of relevant variables is equivalent to appearance of *negative* scaling dimensions. This is a rather unusual phenomenon for $d>0$.

Many common variables are local functions of the shell variables in the vicinity of some shell number n, which we may denote $S(k_n)[U]$. Because of the rescaling involved in the definition of $\mathcal{R}$, no such variable can be a scaling operator in the strict sense. In fact, under the linear map $\mathcal{T}_*$ one such variable is transformed to another variable of that type but with a wave number twice as large: $\mathcal{T}_*\{S(k_n)\}=S'(k_{n+1})$. It is therefore convenient to define a matrix of $\mathcal{T}_*$ in that subspace as

$$T_{\alpha,m;\beta,n}=\frac{\partial\kappa'_\beta(k_{n+1})}{\partial\kappa_\alpha(k_n)} , \tag{36}$$

where the κ's are appropriate sources and the eigenstates of this matrix are scaling operators in a generalized sense that

$$\mathcal{T}_*\{O_\alpha(k_n)\}=2^{-x_\alpha}O_\alpha(k_{n+1}) . \tag{37}$$

Using the fixed point condition and the definition of the scaling operators, it is straightforward to derive the following type of *scaling law* at the fixed point:

$$\langle O_{\alpha_1}(k_{n_1})\cdots O_{\alpha_p}(k_{n_p})\rangle_* = 2^{-(x_{\alpha_1}+\cdots+x_{\alpha_p})}\langle O_{\alpha_1}(2k_{n_1})\cdots O_{\alpha_p}(2k_{n_p})\rangle_* . \tag{38}$$

This is obtained directly by differentiating both sides of the identity $W^*[0,\kappa]=W^*[0,\kappa']$ once with respect to each of the sources $\kappa_{\alpha_1}(k_{n_1}),\ldots,\kappa_{\alpha_p}(k_{n_p})$. This kind of relation has direct phenomenological significance, as well as being theoretically important for the problem of "renormalization" or "statistical continuum limit." It justifies the importance of the notion of scaling variables.

There is a simple relation of the linearized RG map with the correlation functions for the fixed point, which is often exploited in numerical RG works [31]. Suppose $\{S_\beta:\ \beta\in\mathcal{J}\}$ spans an invariant subspace under $\mathcal{T}_*$ as before. The observation required for establishing the relation is that, for calculating the expectation of the variable $S_\alpha[U']$ in the original distribution, either the latter or the one-step RG-transformed distribution may be used. Thus

$$\frac{1}{W_*[0,\kappa]}\left\langle S_\alpha[U']\exp\left[\sum_\beta \kappa_\beta S_\beta[U]\right]\right\rangle_* = \frac{1}{W_*[0,\kappa']}\left\langle S_\alpha[U]\exp\left[\sum_\beta \kappa'_\beta S_\beta[U]\right]\right\rangle_* . \tag{39}$$

Now, by taking the derivative of both sides with respect to κ_β at $\kappa=0$, and noting Eq. (34), it follows that

$$\langle S_\alpha[U']S_\beta[U]\rangle_* - \langle S_\alpha[U']\rangle_*\langle S_\beta[U]\rangle_* = \sum_\gamma (\langle S_\alpha[U]S_\gamma[U]\rangle_* - \langle S_\alpha[U]\rangle_*\langle S_\gamma[U]\rangle_*)T_{\gamma,\beta} . \tag{40}$$

Therefore, by calculating the indicated correlation functions in the fixed-point distribution $P^*_{\{U\}}$, and inverting, one can obtain the linearized RG map (at least in the subspace spanned by $\{S_\beta:\ \beta\in\mathcal{J}\}$). The matrix $T_{\alpha,m;\beta,n}$ may be calculated by the relation generalizing Eq. (40):

$$\langle S_\alpha(k_{n+1})[U']S_\beta(k_m)[U]\rangle_* - \langle S_\alpha(k_{n+1})[U']\rangle_*\langle S_\beta(k_m)[U]\rangle_* = \sum_{\gamma,p}(\langle S_\alpha(k_{n+1})[U]S_\gamma(k_{p+1})[U]\rangle_* - \langle S_\alpha(k_{n+1})[U]_*\langle S_\gamma(k_{p+1})[U]\rangle_*)T_{\gamma,p;\beta,m} . \tag{41}$$

We require for our discussion yet one further generalization of the previous concepts, due to Wegner [29], the idea of scaling fields *away from the critical point.* This concept was originally developed in order to discuss corrections to scaling behavior close to, but slightly away from, the critical point in equilibrium phase transitions. The idea there was to devise an appropriate generalization of Wilson's "eigenoperators" for systems slightly off the critical point. The appropriate generalization was constructed perturbatively in the couplings $\{g_\alpha:\ \alpha\in\mathcal{J}\}$, which measured the distance away from the fixed point, and had the form of variables $O_\alpha(k_n;\{g\})$ with explicit nonlinear dependence upon the variables $\{g\}$. The defining property (to linear order in κ's but nonlinear in g's) was that

$$\mathcal{R}\left\{\sum_i \kappa_i O_{\alpha_i}(k_{n_i};\{g\})\right\} = \sum_i \kappa_i 2^{-x_{\alpha_i}} O_{\alpha_i}(k_{n+1};\{2^{-x}g\}) + O(\kappa^2) , \tag{42}$$

with $\{2^{-x}g\}=\{2^{-x_\alpha}g_\alpha:\ \alpha\in\mathcal{J}\}$. The variables $O_\alpha(k_n;\{g\})$ are the so-called "scaling variables" near the critical point and the associated sources $\kappa_\alpha(k_n;\{g\})$ the "scaling fields" (defined above to linear order). In our development below, we shall *assume* the existence of such scaling variables. Working through the definitions, one can check that the $O_\alpha(k_n;\{g\})$ are specified to linear order as the eigenstates of a matrix $T_{\alpha,m;\beta,n}(\{g\})$, which is obtained from correlation functions as

$$\langle S_\alpha(k_{n+1})[U']S_\beta(k_m)[U]\rangle_{\{g\}} - \langle S_\alpha(k_{n+1})[U']\rangle_{\{g\}}\langle S_\beta(k_m)[U]\rangle_{\{g\}} = \sum_{\gamma,p}\{\langle S_\alpha(k_{n+1})[U]S_\gamma(k_{p+1})[U]\rangle_{\{2^{-x}g\}} - \langle S_\alpha(k_{n+1})[U]\rangle_{\{2^{-x}g\}}\langle S_\gamma(k_{p+1})[U]\rangle_{\{2^{-x}g\}}\}T_{\gamma,p;\beta,m}(\{g\}) . \tag{43}$$

Furthermore, one has a generalized form of the scaling relations, which states that

$$\langle O_{\alpha_1}(k_{n_1},\{g\})\cdots O_{\alpha_p}(k_{n_p},\{g\})\rangle_{\{g\}} = 2^{-(x_{\alpha_1}+\cdots+x_{\alpha_p})}\langle O_{\alpha_1}(2k_{n_1},\{2^{-x}g\})\cdots O_{\alpha_p}(2k_{n_p},\{2^{-x}g\})\rangle_{\{2^{-x}g\}} . \tag{44}$$

Because we generally discuss just a single RG trajectory which corresponds to a specified state with a fixed production mechanism and dissipative cutoff, which uniquely selects the starting PDF for the RG iteration, it is generally unimportant for us to indicate the dependence of the full set of couplings $\{g\}$. (Specifying all of these would fix the particular RG trajectory in occurrence out of the set of all possible ones near the fixed point.) Instead we can indicate the distance from the fixed point by the single parameter $k_0 = L^{-1}$, which is doubled in each RG step. Therefore, our scaling variables are denoted as $O_\alpha(\{k_0\})$, or $O_\alpha(k_n, \{k_0\})$ in the case of a variable locally supported near k_n. Sometimes it useful to explicitly indicate also the uv cutoff k_N, by a superscript, e.g., $O_\alpha^{(N)}(\{k_0\})$. This is particularly helpful when considering the $k_N \to +\infty$ or "continuum" limit.

IV. OPERATOR PRODUCT EXPANSION

A. Canonical scaling of shell polynomials

We now fulfill our promise to derive some general qualitative predictions of the RG methods developed above. Our first result will be a simple application of the theory of the preceding section. We consider *local polynomials* of the shell variables and momentum of the form

$$P_\alpha(k_n)[U] = k_n^s u_n^{p_0} u_{n-1}^{p_1} \cdots u_{n-r}^{p_r} . \tag{45}$$

Notice these are defined so that k_n is the momentum of the highest shell variable in the polynomial. The canonical dimension of this quantity is

$$d_\alpha = s - \tfrac{1}{3} \sum_{i=0}^{r} p_i , \tag{46}$$

in the units determined by $\bar{\epsilon}$. What we show now is that the local polynomials with $n < N$ are scaling variables with Wilson scaling dimension equal to the canonical dimension $x_\alpha = d_\alpha$. (We should actually deal with variables nondimensionalized in terms of $\bar{\epsilon}$ and k_N, the fixed parameters of our RG, but for simplicity we neglect this here.) In other words, there is no "anomalous part" to the scaling dimension. At first sight, this might seem surprising, since it is numerically observed that such variables, e.g., simple moments, show anomalous "multifractal" scaling, i.e., $\langle u_n^p \rangle \sim k_n^{\zeta_p}$ for $k_n \to +\infty$, and ζ_p is *not* the canonical or Kolmogorov scaling dimension [21,22]. This might seem to indicate that the u_n^p have anomalous dimensions, since a general scaling operator $O_\alpha^{(N)}(k_n)$ with scaling dimension x_α obeys the relation

$$\langle O_\alpha(k_n) \rangle_* \sim \left[\frac{k_n}{k_N} \right]^{x_\alpha} \tag{47}$$

for $k_n \ll k_N$ *at the fixed point.* This is a direct consequence of the scaling law. In the continuum or $N \to +\infty$ limit, this goes over into a critical scaling law of the form

$$\langle [O_\alpha(k_n)] \rangle_* \sim k_n^{x_\alpha} , \tag{48}$$

where $[O_\alpha(k_n)] = k_N^{x_\alpha} O_\alpha^{(N)}(k_n)$ is a *renormalized variable* in the continuum theory (with finite expectations for $N \to +\infty$ and dimension x_α in units of inverse length). This might be thought to be the appropriate model of the observed anomalous scaling, but our result proved below shows that this is definitely *not* the correct interpretation.

The proof of the result is very simple and depends upon the observation that

$$P_\alpha(k_{n+1})[U'] = 2^{d_\alpha} P_\alpha(k_n)[U] \tag{49}$$

for $n < N$. This follows directly from the simple structure of our RG for the model, and really makes the result almost obvious. However, we can formally demonstrate the result by observing that

$$\langle P_\alpha(k_{n+1})[U'] P_\beta(k_{m+1})[U'] \rangle_{\bar{\epsilon},L} = \langle P_\alpha(k_{n+1})[U] P_\beta(k_{m+1})[U] \rangle_{\bar{\epsilon},L/2} . \tag{50}$$

We have not used any scaling relation here, but just the definition of the RG transformation. Now, using the Eq. (49) only for the polynomial P_β, it follows that

$$\langle P_\alpha(k_{n+1})[U'] P_\beta(k_m)[U] \rangle_{\bar{\epsilon},L} = \langle P_\alpha(k_{n+1})[U] P_\beta(k_{m+1})[U] \rangle_{\bar{\epsilon},L/2} 2^{-d_\beta} . \tag{51}$$

By the same method of argument applied to one-point expectations, it follows further that

$$\langle P_\alpha(k_{n+1})[U'] \rangle_{\bar{\epsilon},L} \langle P_\beta(k_m)[U] \rangle_{\bar{\epsilon},L} = \langle P_\alpha(k_{n+1})[U] \rangle_{\bar{\epsilon},L/2} \langle P_\beta(k_{m+1})[U] \rangle_{\bar{\epsilon},L/2} 2^{-d_\beta} . \tag{52}$$

Taking the difference of Eqs. (51) and (52) and comparing with Eq. (43), one concludes immediately that in this class of operators

$$T_{\gamma,p;\beta,m}(k_0) = 2^{-d_\beta} \delta_{\gamma,\beta} \delta_{pm} , \tag{53}$$

and from the definitions

$$\mathcal{T}(k_0)\{P_\alpha(k_n)\} = 2^{-d_\alpha} P_\alpha(k_{n+1}) . \tag{54}$$

This is the claimed result. Notice here that it is crucial that $n < N$. In the case $n = N$ the basic relation Eq. (49) fails and such an operator could indeed have an anomalous dimension.

B. A high shell-number expansion and moment scaling laws

In view of this result we must search for an alternative explanation of the observed scaling. In fact, we show that this can be simply understood as the consequence of the *near criticality* of the cascade and the hypothesis of asymptotic completeness. The technical tool we need is a *high-shell-number expansion* analogous to the operator product expansion of field theory. Specializing for simplicity to the shell variable powers, this result says that the powers have the following asymptotic expansion:

$$u_n^p \sim \bar{\epsilon}^{p/3} k_n^{-p/3} \sum_\alpha c_\alpha \, k_n^{-x_\alpha} [O_\alpha(\{k_0\})] , \tag{55}$$

for $k_n \gg k_0 = L^{-1}$ in a weak sense (i.e., in correlation functions of shell variables with fixed momenta). Here we have stated the result in the "continuum" form, as a $k_n \to +\infty$ limit. As before, $[O_\alpha(\{k_0\})]$ is a renormalized variable for the continuum theory which may be defined as a weak limit

$$[O_\alpha(\{k_0\})] \equiv \lim_{N \to +\infty} k_N^{x_\alpha} O_\alpha^{(N)}(\{k_0\}) . \tag{56}$$

The sum in Eq. (55) runs over all such renormalized scaling variables, with c_α some specified expansion coefficients. It is obvious from the form of the expansion that the leading term is from the variable in the sum with the *lowest* dimension. We have already argued that there will be many variables in the shell-model case which have $x_\alpha < 0$, so the leading term is likely to have such a negative dimension in general. The variable with the next lowest dimension gives a subleading correction, and so on. A simple application of the expansion is to the asymptotic evaluation of the shell-variable moments. Using the one-point version of the generalized scaling law Eq. (44), it easily follows that

$$\langle [O_\alpha(\{k_0\})]\rangle_{\bar{\epsilon},L} \sim L^{-x_\alpha} . \tag{57}$$

Now, let $[O_p(\{k_0\})]$ be the variable giving the leading term in the expansion Eq. (55) and x_p its scaling dimension. It follows at once that

$$\langle u_n^p\rangle_{\bar{\epsilon},L} \sim \bar{\epsilon}^{p/3} k_n^{-p/3}(k_n L)^{-x_p} , \tag{58}$$

plus corrections of subleading order as $k_n \to +\infty$. We therefore see that the "multifractal scaling" numerically observed is indeed predicted by the RG method (under all of our assumptions). This result cannot be obtained by dimensional analysis alone, which only gives the weaker result Eq. (15). It is important to note in the above scaling law that the "anomalous scaling" is associated with the finite length scale L in the problem, and it leads to divergence of the moment in the limit $L \to +\infty$. The associated exponent is a scaling dimension of $[O_p(\{k_0\})]$, not of u_n^p. It is also worth mentioning that there is another RG derivation of the Eq. (58) which is more direct for that specific relation than the one presented above [32]. It is based on an RG formulation of Barenblatt's theory of "intermediate asymptotics" [33,14]. That approach presents also some novel calculational strategies at a nonperturbative level and deserves to be pursued, but we do not attempt it here.

Having discussed some implications of the expansion, let us now give its proof. The argument is a simple adaptation of the one used to prove the operator product expansion in field theory and critical phenomena: see the end of Sec. 12.4 in [28] or the related discussion in [34]. We are going to assume that the effective PDF of the shell model for scale k_N is near the fixed point distribution, which requires both that ν be sufficiently small that $k_N\eta \le 1$ and also that L be sufficiently large that $k_N L \gg 1$. We will subsequently take the limit $k_N \to +\infty$, so the result should be valid for the zero-viscosity limiting theory. We must use the properly nondimensionalized variables, which we have avoided up to now:

$$\mathring{u}_n \equiv \frac{u_n}{\bar{\epsilon}^{1/3} k_N^{-1/3}} . \tag{59}$$

We have seen that the simple power $\mathring{u}_n^p$ is, for $n<N$, a scaling variable with scaling dimension equal to $-p/3$, the canonical dimension. Introduce the generating function $W_L[h,\kappa]$ of this variable (we suppress the subscripts $N,\bar{\epsilon}$ on W, since they are fixed under the RG). Now, for a given n, iterate the RG for $N-n$ steps to obtain $W_L[h,\kappa]=W_{L/2^{N-n}}[h,\kappa^{(N-n)}]$, where differentiation of the right-hand side in κ makes one insertion in any correlation function, for the $L/2^{N-n}$ state, of the variable

$$\mathcal{T}^{N-n}\{\mathring{u}_n^p\} = \left[\frac{k_N}{k_n}\right]^{p/3} \mathring{u}_N^p . \tag{60}$$

The use of the linearized transformations is what limits our result to the case $k_n L \gg 1$, since $L/2^{N-n} \gg k_N^{-1}$ must be true for the above equation to be valid. Having iterated down to the cutoff scale, we must now stop because $\mathring{u}_N^p$ is no longer a scaling variable. However, using the assumption of asymptotic completeness, we may expand it into scaling variables weakly in the $L/2^{N-n}$ state, as

$$\mathring{u}_N^p \simeq \sum_\alpha c_\alpha O_\alpha^{(N)}(\{2^{N-n}k_0\}) . \tag{61}$$

Now using the RG transformation again, but in the reverse direction for the generating function of the $O_\alpha^{(N)}$'s, we see that for insertion into arbitrary correlation functions for the original L state, we have the complete (weak) equivalence

$$\mathring{u}_n^p \simeq \left[\frac{k_N}{k_n}\right]^{p/3} \sum_\alpha c_\alpha \left[\frac{k_n}{k_N}\right]^{-x_\alpha} O_\alpha^{(N)}(\{k_0\}) . \tag{62}$$

Reintroducing the dimensional quantity u_n^p, we see that this gives

$$\frac{u_n^p}{\bar{\epsilon}^{p/3} k_N^{-p/3}} \simeq \left[\frac{k_N}{k_n}\right]^{p/3} \sum_\alpha c_\alpha k_n^{-x_\alpha} \cdot k_N^{x_\alpha} O_\alpha^{(N)}(\{k_0\}) . \tag{63}$$

Equation (63) is valid, under the assumptions, for the theory with cutoff k_N. However, dividing both sides of Eq. (63) by the common factor $k_N^{p/3}$ we obtain the result which is valid in the limit $k_N \to +\infty$. This is exactly the claimed expansion result.

It is useful to note that the condition on the viscosity in the above argument is actually that $k_N\eta \le 1$, so we can get a statement for theories with *finite* ν if we allow k_n to range down only to the scale $k_\eta \equiv \eta^{-1}$. We should point out that the condition $k_N\eta \le 1$ is based on the presumption that an effective PDF even for k_N at the Kolmogorov scale $k_N = k_\eta$ is near the fixed point. This seems plausible since we know that in the fixed point theory for cutoff k_N, the effective viscosity goes as $\nu_*^{(N)} \sim \bar{\epsilon}^{1/3} k_N^{-4/3}$, and this is exactly the viscosity for which k_N *is* the Kolmogorov scale wave number. Hence it seems valid to extrapolate the previous results in theories with finite ν down to $k_n = k_\eta$. If that is the case, then we obtain some simple results on dissipation-range scaling. Indeed, we can introduce quantities such as "generalized flatnesses" of the dissipation-range shell variables, such as $F_p \equiv \langle u^p(k_\eta)\rangle_{\bar{\epsilon},L,\nu} / \langle u^2(k_\eta)\rangle_{\bar{\epsilon},L,\nu}^{p/2}$, and the previous results imply that these should scale with Reynolds number $\mathrm{Re} \equiv (k_\eta L)^{4/3}$ as $F_p \sim (\mathrm{Re})^{-3x_p/4 + px_2/8}$ for $\mathrm{Re} \gg 1$. It is especially noteworthy that the exponents x_p appearing here are the same as those in the inertial-range scaling laws.

C. Additive-coupling hypothesis and correlation scaling laws

Further qualitative results can be obtained by the same type of arguments as those above. As one example, let us derive a prediction for scaling of two-point moment correlations in the zero-viscosity limiting theory (or, otherwise, in the inertial range for finite viscosity). The result we establish is that

$$\langle u_n^p u_m^q \rangle_{\bar{\epsilon},L} \sim \bar{\epsilon}^{(p+q)/3} k_n^{-p/3} k_m^{-q/3} \left[\frac{k_n}{k_m} \right]^{-x_p} (k_m L)^{-x_{p+q}} , \tag{64}$$

in the range $L^{-1} \ll k_m \ll k_n$ ($\leq k_\eta$). This is similar to the "Cates-Deutsch" scaling behavior for space correlations of multifractal measures [35]. The proof is quite simple. Employing again the nondimensionalized shell variables, we first use the scaling law Eq. (44), iterating $N-n$ steps until the highest shell variable reaches the cutoff:

$$\langle \mathring{u}_n^p \mathring{u}_m^q \rangle_{\bar{\epsilon},L} = \left[\frac{k_N}{k_n} \right]^{(p+q)/3} \langle \mathring{u}_N^p \mathring{u}_{m+(N-n)}^q \rangle_{\bar{\epsilon},2^{-(N-n)}L} . \tag{65}$$

Now we can use the asymptotic completeness assumption to replace $\mathring{u}_N^p$ by its expansion in Eq. (61), retaining for simplicity only the leading term $O_p^{(N)}(\{2^{N-n}k_0\})$. Subsequently we can iterate by an additional $n-m$ steps, until the second shell variable is at the cutoff scale:

$$\langle \mathring{u}_n^p \mathring{u}_m^q \rangle_{\bar{\epsilon},L} \sim \left[\frac{k_N}{k_n} \right]^{(p+q)/3} \left[\frac{k_n}{k_m} \right]^{-x_p+q/3} \times \langle O_p^{(N)}(\{2^{N-m}k_0\}) \mathring{u}_N^q \rangle_{\bar{\epsilon},2^{-(N-m)}L} . \tag{66}$$

Now we can replace $\mathring{u}_N^q$ by the leading term in its expansion in scaling variables $O_q^{(N)}(\{2^{N-m}k_0\})$. We then encounter the product $O_p^{(N)}(\{2^{N-m}k_0\})O_q^{(N)}(\{2^{N-m}k_0\})$, which is no longer a scaling variable but can itself be expanded in terms of scaling variables. It is very plausible that the leading term in this expansion (the one with the smallest dimension) is just $O_{p+q}^{(N)}(\{2^{N-m}k_0\})$. The reason is that all the terms which appear in the expansion of $O_p^{(N)}(\{2^{N-m}k_0\})O_q^{(N)}(\{2^{N-m}k_0\})$ are also terms which could appear in the expansion of $\mathring{u}_N^{p+q}$, in view of all the symmetries and invariances of the theory. But the leading term in the expansion of that variable is exactly $O_{p+q}^{(N)}(\{2^{N-m}k_0\})$. It would be something of an accident for this term to have zero coefficient in the expansion of $O_p^{(N)}(\{2^{N-m}k_0\})O_q^{(N)}(\{2^{N-m}k_0\})$. Without attempting any rigorous proof of this "additive coupling," let us explore its consequences as a plausible hypothesis. In that case, we have the result

$$\langle \mathring{u}_n^p \mathring{u}_m^q \rangle_{\bar{\epsilon},L} \sim \left[\frac{k_N}{k_n} \right]^{(p+q)/3} \left[\frac{k_n}{k_m} \right]^{-x_p+q/3} \times \langle O_{p+q}^{(N)}(\{2^{N-m}k_0\}) \rangle_{\bar{\epsilon},2^{-(N-m)}L} . \tag{67}$$

Now iterating the scaling law by $N-m$ steps in the reverse order, we obtain

$$\langle \mathring{u}_n^p \mathring{u}_m^q \rangle_{\bar{\epsilon},L} \sim \left[\frac{k_N}{k_n} \right]^{(p+q)/3} \left[\frac{k_n}{k_m} \right]^{-x_p+q/3} \times \left[\frac{k_N}{k_m} \right]^{x_{p+q}} \langle O_{p+q}^{(N)}(\{k_0\}) \rangle_{\bar{\epsilon},L} . \tag{68}$$

Finally, returning to the dimensional shell variables, we obtain, after canceling factors, that

$$\langle u_n^p u_m^q \rangle_{\bar{\epsilon},L} \sim \bar{\epsilon}^{(p+q)/3} k_n^{-p/3} k_m^{-q/3} \left[\frac{k_n}{k_m} \right]^{-x_p} \times k_m^{-x_{p+q}} \langle [O_{p+q}(\{k_0\})] \rangle_{\bar{\epsilon},L} , \tag{69}$$

which gives exactly the claimed Eq. (64).

D. Renormalization group and multiplicative cascade models

The previous results show how RG, when supplemented with some additional plausible assumptions, which may be generally true or special to the particular situation, can yield results that cannot be obtained by dimensional analysis alone. In particular, the power-law correction to the moment scaling in Eq. (58) is not predicted by DA alone. There is a traditional expectation of such types of corrections to Kolmogorov scaling which is based on simple *cascade ansatz,* which model the transport of energy to small scales by random multiplicative processes. Such types of models go back to the earliest attempts of Kolmogorov and Obukhov to make corrections to K41 which would take into account the Landau objection [36,37]. The 1962 "log-normal model" and the later generalizations in a wide class of "multifractal" cascade models by Novikov [38], Mandelbrot [39], and others all predict power-law corrections of the indicated type. In fact, all of the previous derived results for our model can be simply derived from such cascade ansatz. Within such an ansatz, the relation like our Eq. (58) follows, with the exponent related to the random multiplier M as $-x_p = \log_2 \langle M^p \rangle$, where $\langle\ \rangle$ indicates expectation with respect to the multiplier distribution (e.g., see [40]). Furthermore, the result like Eq. (64) can also be established on the basis of such an ansatz. Heuristically, it may be interpreted as follows: the moments propagate together from scale $k_0 = L^{-1}$ to the scale k_m, leading to the factor $(k_m L)^{-x_{p+q}}$, and thereafter the higher-shell moment propagates alone from scale k_m to scale k_n, leading to the factor $(k_n/k_m)^{-x_p}$. An analytic calculation verifies this result. Therefore, the RG results are substantially the same as those of the simpler cascade ansatz. The difficulty with such ansatz is that they make a too specific model assumption about the nature of the energy transport, e.g., they assume commonly no backward energy transfer, a completely stochastic distribution process, etc. It is not clear how they may be made consistent with the dynamical equations of motion. Therefore, one cannot really be confident of the predicted form of the scaling corrections. In contrast, the RG argument requires no model assumptions about the energy transport and is in a framework completely compatible with the equations of motion. It incorporates some of the same "cascade" ideas primarily through the assumption that there should be negative scaling dimensions, associated to the many relevant directions leaving the fixed point. The justification for that goes back ultimately to the idea of the "local cascade" in which individual steps are roughly independent, leading to loss of information of the large scales. However, we would argue that the RG argument, while based on some of the same physical ideas, actually makes less specific assumptions and therefore gives a considerably better ground to the predicted relations.

V. QUANTITATIVE METHODS

A. What can be calculated?

In the final analysis, the RG methodology in turbulence problems should be judged on the basis of whether it can actually yield useful computational strategies for quantities of interest, e.g., scaling exponents. While this has proved to be the case for many equilibrium problems, it is essentially untested for turbulence calculation. The main difficulty facing the application of RG to turbulence—just as faces any other method—is the essential strong nonlinear coupling of the problem. RG in no way ameliorates this difficulty. However, there are some methods which have been used in equilibrium problems of strong-coupling nature, in conjunction with RG strategies, which have been quite successful. One of these is *numerical implementation* of RG and the other is *perturbation expansion* based on a "hidden" small parameter. We discuss each of these possibilities in turn, specifically for the context of the shell model.

B. Numerical RG

The numerical application of RG to calculating scaling exponents in equilibrium critical systems is primarily based on the relation Eq. (40) between correlation functions and linearized RG map. This is the method of *Monte Carlo renormalization group* first proposed by Ma [41] (see also [31]). In this method, a numerical algorithm, such as the Metropolis method, is used to generate an ensemble of spin configurations with the probabilities of the equilibrium distributions. If one can fix the parameters of the distribution to be at the fixed point initially (e.g., by using exact results from duality, etc.), then the required correlations can be calculated directly. Otherwise, the fixed-point distribution can be obtained by performing the RG transformation in terms of "block spins" directly on the generated spin configurations. Close to the critical point several iterations are usually sufficient to drive the ensemble distribution to the fixed point. Once the fixed-point correlation functions are known, the linearized RG map is obtained by inversion and both the scaling variables and exponents obtained as its eigenvectors and eigenvalues.

The same type of strategy might be applied in the case of turbulence. A basic difficulty is how to calculate the correlation functions. Unfortunately, the only way it is really known how to obtain them is by direct numerical simulation of the dynamics. In that case, it is obviously much easier to calculate the exponents directly from a log-log plot of the momenta $\langle u_n^p \rangle$ versus k_n. The method of calculating two-point correlations of a large class of operators, inverting to obtain the RG map, and then diagonalizing is obviously much less efficient (and probably less accurate). As emphasized by Kraichnan [11], a statistical-mechanics method which requires more work than direct simulation is obviously not the one we want. The success of a numerical RG method depends partly upon developing a good method to obtain the correlation functions, short of direct simulation of the dynamics. One possibility here might use the existence of the path-integral representation Eq. (16). There are already approaches to quantum Monte Carlo based upon applying the Monte Carlo methods, originally developed for equilibrium Gibbs distributions described by a Hamiltonian, to the Feynman path-integral representation of quantum wave functions for a specified classical Lagrangian (e.g., [42] and references therein). Such methods might be applied to the MSR path-integral representation. Another possibility is based upon the "decimation theory" ideas of Kraichnan [11]. In this approach, the number of degrees of freedom in the exact dynamics is reduced, based upon statistical redundancy of modes, and the eliminated degrees of freedom replaced by suitable Langevin forces subject to imposed constraints from the exact dynamics. In the case of the shell model, such a strategy is not obviously useful because it is already strongly "decimated." (In fact, we have been looking at the shell dynamics as a model problem, but, since they are sufficiently simple for computer simulation, they might instead be regarded as the solution. A strongly decimated version of the Navier-Stokes equation in a wavelet representation might well resemble a shell model with additional Langevin constraint forces.) However, the correlation functions in time should have smoothness properties which are not present in the individual realizations of the shell dynamics, which show strong temporal intermittency. This might be made the basis of a decimation approach in time. Still, a basic objection to all these ideas is the following: Why not just calculate the moments directly? One answer is that the RG method gives more information, e.g., the scaling variables themselves. Another interesting possibility of the RG method is based upon finding the "fixed point subgrid model." This is certainly likely to be a very complicated dynamical object, with all types of higher-order nonlinearities, non-Markovian features, etc. However, it incorporates information about the buildup of intermittency in infinitely many cascade steps. If it can somehow be found, and then simulated, information could be obtained in a relatively small-scale computation that would otherwise only be obtainable from a simulation at extremely high Reynolds number.

C. Perturbation theory

The second possibility for quantitative calculation is perturbation expansion. Since the model of interest is strongly coupled, this strategy depends upon inventing a sequence of models, depending upon a parameter, which interpolate between the original strongly coupled model and a soluble limit. Then, an expansion can be made in the interpolation parameter around the soluble limit and the results finally extrapolated to the original strongly coupled case. Examples of this for equilibrium theory are the ϵ and $1/N$ expansions. These methods have proved especially useful for clarifying the qualitative nature of phase diagrams, but they have also proved in many cases to give accurate quantitative results, especially when improved by resummation techniques.

To apply this method in turbulence problems requires firstly the invention of suitable interpolation problems. As emphasized by Kraichnan, the intermittency corrections depend crucially upon the exact form of the Navier-Stokes nonlinearity, so that any approach to calculating the scaling exponents must correctly treat the nonlinearity [43]. The same type of arguments apply to the shell models. This is an inherent problem for the perturbation method since the interpolation problems necessarily modify the nonlinearity. The most natural approach may be to expand around the direct interaction approximation (DIA) solution. The DIA equations are derived in an $N\rightarrow+\infty$ limit from the random coupling model (RCM), which employs N copies of the original problem coupled together in collective coordinates with random dynamical phases [44]. $N=1$ is the original problem itself. (Another large-N approach to the DIA is a "spherical model" based upon using large-spin representations of the rotation group [45].) Since the interaction strengths in the RCM are the same as for the original problem, only with random phases, some important features of the nonlinearity of the original problem are preserved in the limit. The most straightforward thing to attempt is therefore to expand in $1/N$ around the DIA solution, either for the original RCM or the spherical model. Some other ideas for expansion around the DIA are discussed in [46].

The other main method used in critical phenomena, the ϵ expansion, has already been attempted for turbulence [2,3]. However, this method has a number of severe problems. The ϵ expansion which is presently developed is for a model of power-law randomly stirred fluids, in which the force is over the whole inertial range. Even for $\epsilon=4$, where a $\frac{5}{3}$ law is dimensionally predicted, one is not really studying turbulence but only a "turbulencelike" system. However, the method has other serious difficulties, which we hope to discuss elsewhere at more length. For one thing, the RG for these models turns out not to be systematic, even for small ϵ, because of an infinite number of marginal variables. This may be connected with the presence of an infinite number of fixed points, since the variables in question are unlikely to get loop corrections to the marginal scaling. Also, it is very likely that an important crossover occurs in those models at $\epsilon=3$, so it is then useless to extrapolate from small ϵ to $\epsilon=4$. In the shell models, there is a further problem, since the RG recursion is modified—due to the fact that eddy viscosity is a strictly cutoff scale effect—and the problem is not weakly coupled even for small ϵ. In our opinion, the present ϵ expansion is not likely to be very useful in turbulence theory.

Whatever expansion technique might be devised, RG is an important tool in systematically improving the perturbation expansion. Despite the ultimate triviality of the fixed-point condition Eq. (19), one should appreciate that it is *nonperturbative* in origin, and its consequences, like the power-law scaling Eq. (58), need not be true order by order in the perturbation expansion. Indeed, the way power laws typically appear in perturbation expansion is as *logarithmic divergences,* and the RG obtains power laws perturbatively by performing an infinite resummation of terms at all orders in the naive expansion, the so-called leading-logarithm series (e.g., see [47], Sec. 7.6). Therefore, the RG invariance and fixed-point condition are very important tools especially in perturbation approaches, since they are a way of incorporating exact nonperturbative results which are otherwise missed by the expansions.

ACKNOWLEDGMENTS

I wish to thank J. Eggers, U. Frisch, N. Goldenfeld, R. H. Kraichnan, Y. Oono, and Z.-S. She for conversations on the subject of this work, particularly Professor Kraichnan for his illuminating remarks on statistical-mechanical approaches to the "turbulence problem." I am very grateful also to N. Goldenfeld and Y. Oono for their many constructive remarks on the manuscript and also for their own works and point of view on RG, which have influenced mine. Funding from NSF Grant No. DMR-90-15791 and the Exxon Education Foundation is gratefully acknowledged.

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